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Equality of Measure and Topological Entropies for Cellular Automata

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Abstract. In the paper we consider cellular automata defined on metric spaces $(A^{\mathbb{N}}, d)$ or $(A^{\mathbb{Z}}, d)$ endowed with the uniform Bernoulli measure μ and present general constructions of such automata which are surjective, not positively expansive and μ is the measure of the maximal and positive entropy.

Keywords: infinite word, uniform Bernoulli measure, surjective cellular automata, entropy, dynamical system.

1. Introduction

Cellular automata can be characterized as homomorphisms of a metric space of one or bi-infinite words defined over a finite alphabet A. In this convention cellular automata define continuous transformations which are μ-measurable and preserve the measure for some probabilistic measure μ. Hence it is possible to consider them as dynamical systems and investigate topological entropy and measure-theoretic entropy of these systems (or transformations). L.W. Goodwyn in [5] proved that for such transformations topological entropy is greater than or equal to its metrical entropy. Usually these two types of entropies are considered as some measure of dynamical complexity of the transformation. Among the others, there is one interesting case if these two entropies are equal. There are examples of dynamical systems for which a probabilistic measure μ such that these two entropies are equal does not exist. If the equality is true for some μ we say that μ is the measure of the maximal entropy.

F. Blanchard, A. Maass [3] proved that a uniform Bernoulli measure μ associated with any positively expansive cellular automaton defined on the metric space of one-sided infinite words $A^{\mathbb{N}}$ is the measure of the maximal entropy. It is known [2, 7] that any cellular automaton is continuous and that any surjective automaton preserves the uniform Bernoulli measure. We undertake the problem considered by F. Blanchard, A. Maass [3] getting rid of the positive expansivity assumption. In the paper we present general constructions of cellular automata which are surjective, not positively expansive and the uniform Bernoulli measure μ is the measure of the maximal and positive entropy.

2. Preliminaries

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the sets of non-negative integers, integers and real numbers, respectively. For any set Y denote by $\#Y$, its cardinality. Let us assume that A is a finite set such that $\#A \geq 2$. A non-empty and finite word over (an alphabet) A is a function defined on a discrete interval $[0, k]$, where $k \in \mathbb{N} \setminus \{0\}$, with values in A. The set of all words over A with catenation of words forms a free semigroup $(A^+, \cdot)$ over the alphabet A. The length $|u|$ of a word $u \in A^+$ is defined to be the cardinality of the domain of u. The set of all words in A^+ of the length equal to $n \in \mathbb{N} \setminus \{0\}$ is denoted by A^n. Let us denote a neutral element of catenation by 1 and call it the empty word. Adding the empty word to A^+ we obtain a free monoid $A^* = (A^+ \cup \{1\}, \cdot)$. By the definition the length of the empty word 1 is 0. One-sided or two-sided infinite words over A are functions defined on discrete intervals $[0, \infty) = \mathbb{N}$, or $(-\infty, \infty) = \mathbb{Z}$, taking values in A. The sets of all such infinite words are denoted by $A^{\mathbb{N}}$, $A^{\mathbb{Z}}$, respectively. We will consider also words defined on finite intervals of the type $I = [i, j]$ where $i < j \in \mathbb{Z}$. For two discrete intervals I, J such that $J \subset I$ and for a word u defined on I we denote the restriction of u to the interval J by u_J.

Put $\mathbb{X} = \mathbb{Z}$, or $\mathbb{X} = \mathbb{N}$. Let $x, y \in A^{\mathbb{X}}$ and define $d: A^{\mathbb{X}} \times A^{\mathbb{X}} \to \mathbb{R}$ by $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x(0) \neq y(0)$, and for all other cases

$$d(x,\,y) = \begin{cases} 2^{-(i+1)},\ i = max\{j \geq 0 : x_{[-j,\,j]} = y_{[-j,\,j]}\}\ if\ \ \mathbb{X} = \mathbb{Z}, \\ 2^{-(i+1)},\ i = max\{j \geq 0 : x_{[0,\,j]} = y_{[0,\,j]}\}\ if\ \ \mathbb{X} = \mathbb{N}. \end{cases}$$

The function d is a metric on $A^{\mathbb{X}}$ and clearly $(A^{\mathbb{X}},\,d)$ is a compact topological (metric) space. For $x \in A^{\mathbb{X}}$ and $r > 0\,(r \in \mathbb{R})$ an open ball with the center at x and radius r is the set $K(x,\,r) = \{y \in A^{\mathbb{X}} : d(x,\,y) < r\}$. The family of all such balls $\alpha = \{K(x,\,r) :\ x \in A^{\mathbb{X}},\ r \in \mathbb{R},\ r > 0\}$ is the base of the topology τ_d defined by the metric d. A σ-algebra of Borel sets generated by τ_d is denoted by $\beta(A^{\mathbb{X}})$. On the σ-algebra $\beta(A^{\mathbb{X}})$ we define a uniform Bernoulli measure μ, putting for any ball with the center at $x \in A^{\mathbb{X}}$ and radius $r = 2^{-n}$ $(n \in \mathbb{N})$:

$$\mu(K(x,\,2^{-n})) = \begin{cases} \#A^{-(2n+1)}\,if\ \ \mathbb{X} = \mathbb{Z}, \\ \#A^{-(n+1)}\,if\ \ \mathbb{X} = \mathbb{N}. \end{cases}$$

A dynamical system is a pair (Y,T), where Y is a compact topological space and $T : Y \to Y$ is a continuous mapping. If $Y = A^{\mathbb{X}}$ then $(A^{\mathbb{X}},T)$ is said to be a symbolic dynamical system. Considering dynamical systems one can use as a research tool the notion of a topological entropy. Having a T-invariant, probabilistic measure ν it is also possible to consider a measure-theoretic entropy of dynamical systems. These entropies can be interpreted as some measures of a chaotic character of systems. A measure ν for which topological and measure-theoretic entropies are equal is said to be the measure of maximal entropy.

In the paper we focus on a topological and measure-theoretic entropy of symbolic dynamical systems which are defined by cellular automata. To obtain a better transparency of the problem we restrict our presentation to cellular automata. All the obtained results could be reworded in the language of dynamical systems and connections with symbolic dynamical systems are easy to establish.

Now we define a one dimensional cellular automaton CA. It is a mapping $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ defined as follows. Let us fix a non-negative integer $k \geq 0$ and assume that F' is a function defined locally on $A^{\mathbb{X}}$ that is for any $x \in A^{\mathbb{X}}$ if there exists $i \in \mathbb{X}$ such that $x_{[0,\,k]}(j) = x(i+j)$ for any $j \in [0,\,k]$, then $F'(x_{[0,\,k]}) = b = F'(x_{[i+0,\,i+k]})$ for some $b \in A$. Hence to compute a value of F' it is enough to know all its values on words of A^{k+1}. We can actually consider A^{k+1} as a domain of F'. Now let us fix numbers $m,\,a \in \mathbb{X}$ such that $m \leq a$ and put $k = a - m$. A mapping $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ such that for any $x \in A^{\mathbb{X}}$ and $i \in \mathbb{X}$ we have $F(x)(i) = F'(x_{[i+m,\,i+a]})$ is called a (one dimensional) cellular automaton. The integer k is called the diameter of CA and the mapping F' is called a local rule. We say that a cellular automaton is two-sided (one-sided) if $\mathbb{X} = \mathbb{Z}$ ($\mathbb{X} = \mathbb{N}$). The defined mapping F is continuous [2, 7].

Now we remind two definitions of entropy. Let $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ be a cellular automaton. An open cover of a topological space $(A^{\mathbb{X}}, d)$ is a family V of open sets whose union is $A^{\mathbb{X}}$. Let $O_{A^{\mathbb{X}}}$ be a class of all open covers of $(A^{\mathbb{X}}, d)$ and $U, V \in O_{A^{\mathbb{X}}}$. We say that U is a subcover of V if $U \subset V$. For any cover $V \in O_{A^{\mathbb{X}}}$ we put $|V| = min\{\#W : W \text{ is a finite subcover of } V\}$ and call it the size of V. Intersection of two open covers $U, V \in O_{A^{\mathbb{X}}}$, denoted by $U \vee V$ is a family $U \vee V = \{P \cap Q : P \in U, Q \in V\}$. For any $V \in O_{A^{\mathbb{X}}}$, we define n-th power of V as a family $V^n = V \vee F^{-1}(V) \vee ... \vee F^{-(n-1)}(V)$. Topological entropy $h(A^{\mathbb{X}}, F)$ of a cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is

$$h(A^{\mathbb{X}}, F) = sup\{H(A^{\mathbb{X}}, F, V) : V \in O_{A^{\mathbb{X}}}\},$$

where $H(A^{\mathbb{X}}, F, V) = lim_{n\to\infty}\frac{\log|V^n|}{n}$.

Surjective cellular automata preserve a uniform Bernoulli measure ([2, 7, 8]). Thus the uniform Bernoulli measure μ is F-invariant and it is possible to define a measure-theoretic entropy for F. Hence we consider a topological space $A^{\mathbb{X}}$ endowed with the uniform Bernoulli measure μ defined on the σ-algebra of Borel sets $\beta(A^{\mathbb{X}})$. A finite family of Borel sets $V = \{Q_i \in \beta(A^{\mathbb{X}})\}_{i\in I}$, is called a finite partition of $A^{\mathbb{X}}$ if the following conditions are fulfilled:

1. $\bigcup_{i\in I} Q_i = A^{\mathbb{X}}$,
2. if $i, j \in I$, $i \neq j$ then $Q_i \cap Q_j = \emptyset$.

Entropy of a finite partition $V = \{Q_i\}_{i\in I}$ is defined by

$$H(V) = -\sum_{i\in I} \mu(Q_i)\log\mu(Q_i), \quad (\text{if } \mu(Q_i) = 0, \text{ then } 0 \log 0 = 0).$$

Let $O'_{A^{\mathbb{X}}}$ denote a class of all finite partitions of $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}))$. We define for any $U, V \in O'_{A^{\mathbb{X}}}$ intersection of two partitions $U \vee V$ and n-th power of V analogously as for open covers.

A measure-theoretic entropy $h(F)$ of a cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is

$$h(F) = sup\{H(F, V) : V \in O'_{A^{\mathbb{X}}}\},$$

where $H(F, V) = lim_{n\to\infty}\frac{1}{n}H(V^n)$.

Let $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ be a cellular automaton. We associate with F, in a natural way, its symbolic dynamical system $(A^{\mathbb{X}}, F)$. If F is surjective and μ is the uniform Bernoulli measure then it is denoted in the sequel $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, F)$.

3. Basic facts

In this section we collect some basic notions and facts connected with one dimensional cellular automata and their topological and measure-theoretic entropies.

Theorem 1. *(G. A. Hedlund [7]) A mapping $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is a one dimensional cellular automaton if and only if it is continuous and commutes with the shift mapping $\sigma : A^{\mathbb{X}} \to A^{\mathbb{X}}$, that is $\sigma \circ F = F \circ \sigma$.*

Corollary 1. *Just from the continuity of $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ and the compactness of $(A^{\mathbb{X}}, d)$ follows that $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is uniformly continuous. This property implies that for any cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ there exists a number $r \in \mathbb{N}$ such that for any $x \in A^{\mathbb{X}}$ and $i \in \mathbb{X}$ we have*

$$F(x)(i) = \begin{cases} F'(x_{[i-r,\, i+r]}), & F' : A^{2r+1} \to A \ if \ \ \mathbb{X} = \mathbb{Z}, \\ F'(x_{[i,\, i+r]}), & F' : A^{r+1} \to A \ if \ \ \mathbb{X} = \mathbb{N}. \end{cases}$$

Let us assume that $k \in \mathbb{N}$ and $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is a cellular automaton with the local rule $F' : A^{k+1} \to A$. If for any fixed word $u \in A^*$ the discrete interval $[0, |u| - k)$ is not empty, then for every $i \in [0, |u| - k)$ we put $F'(u)(i) = F'(u_{[i,\, i+k]})$. If the interval is empty, then we put $F'(u) = 1$. The described procedure extends $F' : A^{k+1} \to A$ to the mapping $F' : A^* \to A^*$.

Theorem 2. *(G.A. Hedlund [3, 7]) A cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ with a local rule $F' : A^{k+1} \to A$ is surjective if and only if for any word $u \in A^+$, $\#F'^{-1}(u) = \#A^k$.*

From the above theorem and Corollary 1 one could derive the following theorem. Below we present the proof of this statement which is a bit different from the original one (see for example [7]).

Theorem 3. *([2, 7, 8]) If a cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is surjective then it preserves a uniform Bernoulli measure μ.*

Proof. In view of Corollary 1 for any cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ there exists $r \in \mathbb{N}$ such that for any $i \in \mathbb{X}$ we have

$$F(x)(i) = \begin{cases} F'(x_{[i-r,\, i+r]}), & F' : A^{2r+1} \to A \ if \ \ \mathbb{X} = \mathbb{Z}, \\ F'(x_{[i,\, i+r]}), & F' : A^{r+1} \to A \ if \ \ \mathbb{X} = \mathbb{N}. \end{cases}$$

Theorem 2 implies that a cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is surjective if and only if for any word $u \in A^+$, $\#F'^{-1}(u) = \#A^k$, where $k = 2r$ for the case (i) $\mathbb{X} = \mathbb{Z}$ and $k = r$ for the case (ii) $\mathbb{X} = \mathbb{N}$.

Hence for the case

(i) we have $\mu(F^{-1}(K(x, 2^{-n}))) = \frac{\#A^{2r}}{\#A^{2n+1+2r}} = \#A^{-(2n+1)} = \mu(K(x, 2^{-n}))$ and for

(ii) $\mu(F^{-1}(K(x, 2^{-n}))) = \mu(\bigcup_{z \in F'^{-1}(x_{[0,\,n]})} \{y \in A^{\mathbb{N}} :\ y_{[0,\,n+r]} = z\}) = \sum_{z \in F'^{-1}(x_{[0,\,n]})} \mu(\{y \in A^{\mathbb{N}} :\ y_{[0,\,n+r]} = z\}) = \frac{\#A^{r}}{\#A^{n+1+r}} = \#A^{-(n+1)} = \mu(K(x, 2^{-n}))$, where $n \in \mathbb{N}$.

Note that the family $\alpha = \{K(x, r) :\ x \in A^{\mathbb{X}},\ r \in \mathbb{R},\ r > 0\}$ is the base of the topology τ_d and that any two balls in $(A^{\mathbb{X}}, d)$ are disjoint or one is contained in the other. Hence any non-empty and open set in $(A^{\mathbb{X}}, \tau_d)$ is a sum of a countable family of pairwise disjoint balls. In view of a countable additivity of μ, it implies that if $P \in \tau_d$, then $\mu(F^{-1}(P)) = \mu(P)$. Since $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is continuous we have $F^{-1}(Q) \in \beta(A^{\mathbb{X}})$ for any $Q \in \beta(A^{\mathbb{X}})$. Moreover, the measure μ is a Borel probabilistic and regular measure on $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}))$ [8]. It means that for any $Q \in \beta(A^{\mathbb{X}})$ and $\varepsilon > 0$ there exist open sets $U, V \in \tau_d$ such that $K = A^{\mathbb{X}} \setminus V$, $K \subset Q \subset U$ and $\mu(U) \setminus \mu(K) = \mu(U \setminus K) < \varepsilon$.

We have also

$$\mu(F^{-1}(K)) = \mu(F^{-1}(A^{\mathbb{X}} \setminus V)) = \mu(F^{-1}(A^{\mathbb{X}})) \setminus \mu(F^{-1}(V)) = \mu(A^{\mathbb{X}}) \setminus \mu(V) = \mu(A^{\mathbb{X}} \setminus V) = \mu(K).$$

The fact $K \subset Q \subset U$, implies

$$F^{-1}(K) \subset F^{-1}(Q) \subset F^{-1}(U), \text{ and } \mu(K) = \mu(F^{-1}(K)) \leq \mu(F^{-1}(Q)) \leq \mu(F^{-1}(U)) = \mu(U).$$

Thus $\mu(F^{-1}(Q)) = \mu(Q)$ for any $Q \in \beta(A^{\mathbb{X}})$ what finishes the proof. □

A cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ with a local rule $F' : A^{k+1} \to A$ is said to be left-permutative (right-permutative) if for any $u \in A^k$, and $b \in A$, there exists exactly one $a \in A$ such that $F'(au) = b$ ($F'(ua) = b$)).

Theorem 4. *([7]) Any left-permutative (right-permutative) cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is surjective.*

To illustrate the theorem we present the following example.

Example 1. *Over an alphabet $A = \{0, 1\}$ we define, by local rules I', σ' and f', three cellular automata:*

(a) $I : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$,
$I'(000) = I'(001) = I'(100) = I'(101) = 0$,
$I'(010) = I'(011) = I'(110) = I'(111) = 1$ *and* $I(x)(i) = I'(x_{[i-1,\,i+1]})$ *for* $x \in A^{\mathbb{Z}}$ *and* $i \in \mathbb{Z}$.
The automaton $I : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is not left-permutative nor right-permutative but is surjective since $I : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is an identity mapping.

(b) $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$,
$\sigma'(000) = 0$, $\sigma'(010) = 0$, $\sigma'(100) = 0$, $\sigma'(110) = 0$,
$\sigma'(001) = 1$, $\sigma'(011) = 1$, $\sigma'(101) = 1$, $\sigma'(111) = 1$ *and* $\sigma(x)(i) = \sigma'(x_{[i-1,\,i+1]})$ *for* $x \in A^{\mathbb{Z}}$ *and* $i \in \mathbb{Z}$.

The automaton $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is right-permutative, not left-permutative and from the Theorem 4 is surjective. Note that* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is a shift mapping, that is* $\sigma(x)(i) = x(i+1)$.

(c) $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$,
$f'(000) = f'(011) = f'(101) = f'(110) = 1$,
$f'(001) = f'(010) = f'(100) = f'(111) = 0$ *and* $f(x)(i) = f'(x_{[i-1,\,i+1]})$ *for* $x \in A^{\mathbb{Z}}$ *and* $i \in \mathbb{Z}$.
The automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is left-permutative and right-permutative and from the Theorem 4 surjective.*

Note that left or right permutativity of a cellular automaton $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ depends on the definition of a local rule $F' : A^{k+1} \to A$. Consider the automaton $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ from the above example. This automaton is defined also by the function (new local rule) $\sigma' : A^5 \to A$ such that $\sigma'(uba) = b$ for any $u \in A^3$, $a, b \in A$ and $\sigma(x)(i) = \sigma'(x_{[i-2,\,i+2]})$ for any $x \in A^{\mathbb{Z}}, i \in \mathbb{Z}$. Now $\sigma'(00010) = 1$, $\sigma'(00011) = 1$ and if $u = 0001$, $b = 1$, then $\sigma'(u0) = b$, $\sigma'(u1) = b$, what implies that $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is not right-permutative. Of course all the properties of the cellular automaton $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ are independent of the form of a local rule that defines it.

Let us assume that $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ is a cellular automaton. The function F is said to be

1. Equicontinuous at the point $y \in A^{\mathbb{X}}$, if and only if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for every $x \in K(y, \delta)$, and $n \in \mathbb{N}$, we have $d(F^n(x), F^n(y)) < \varepsilon$. If F is equicontinuous at any point $y \in A^{\mathbb{X}}$, then F is said to be equicontinuous.

2. Sensitive to initial conditions if and only if there exists $\varepsilon > 0$, such that for any $y \in A^{\mathbb{X}}$, and any $\delta > 0$ there exist $x \in K(y, \delta)$, $n \in \mathbb{N}$, for which $d(F^n(x), F^n(y)) \geq \varepsilon$.

3. Positively expansive if and only if there exists $\varepsilon > 0$, such that for any $x, y \in A^{\mathbb{X}}$, $y \neq x$, there exists $n \in \mathbb{N}$, for which $d(F^n(x), F^n(y)) \geq \varepsilon$.

The following theorem gives a classification of cellular automata taking into account their dynamical properties.

THEOREM 5. *([6]) Any cellular automaton* $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$ *have one of the following properties:*

1. F is equicontinuous,

2. there exist points of equicontinuity of F but it is not equicontinuous,

3. F is sensitive to initial conditions but not positively expansive,

4. F is positively expansive.

Positively expansive cellular automata $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ have the following property.

THEOREM 6. *(F. Blanchard, A. Maass [3])*
If $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a positively expansive cellular automaton, then F is surjective and $h(A^{\mathbb{N}}, F) = h(F) = \log k$ for $k \in \mathbb{N} \setminus \{0\}$.

We will use in the sequel the following version of a cartesian product of two symbolic dynamical systems $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, F)$ and $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, G)$ associated with surjective cellular automata $F : A^{\mathbb{X}} \to A^{\mathbb{X}}$, and $G : A^{\mathbb{X}} \to A^{\mathbb{X}}$. A dynamical system $(A^{\mathbb{X}} \times A^{\mathbb{X}}, F \times G)$ such that $F \times G : A^{\mathbb{X}} \times A^{\mathbb{X}} \to A^{\mathbb{X}} \times A^{\mathbb{X}}$ is a function and

$$(F \times G)(x, y) = (F(x), G(y)) \in A^{\mathbb{X}} \times A^{\mathbb{X}}, \text{ for } (x, y) \in A^{\mathbb{X}} \times A^{\mathbb{X}},$$

is called the cartesian product of $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, F)$ and $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, G)$.

If we consider topological aspects of $(A^{\mathbb{X}} \times A^{\mathbb{X}}, F \times G)$, then we define a metric space $(A^{\mathbb{X}} \times A^{\mathbb{X}}, d')$ putting for any $z = (x, y)$, $z' = (x', y') \in A^{\mathbb{X}} \times A^{\mathbb{X}}$, $d'(z, z') = max\{d(x, x'), d(y, y')\}$. If we consider measure aspects of $(A^{\mathbb{X}} \times A^{\mathbb{X}}, \beta(A^{\mathbb{X}}) \times \beta(A^{\mathbb{X}}), \nu = \mu \times \mu, F \times G)$, then we define a probabilistic measure $\nu = \mu \times \mu$ on a σ-algebra $[\beta(A^{\mathbb{X}}) \times \beta(A^{\mathbb{X}})]$ generated by the family $\beta(A^{\mathbb{X}}) \times \beta(A^{\mathbb{X}})$ putting $\nu(P \times Q) = \mu(P) \cdot \mu(Q)$ for any $P \times Q \in \beta(A^{\mathbb{X}}) \times \beta(A^{\mathbb{X}})$. Note that a cartesian product of two symbolic dynamical systems associated with cellular automata is, in general a dynamical system.

THEOREM 7. *(compare [4, 7])*
Let $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, F)$ and $(A^{\mathbb{X}}, \beta(A^{\mathbb{X}}), \mu, G)$ be two symbolic dynamical systems associated with surjective cellular automata F, G and $(A^{\mathbb{X}} \times A^{\mathbb{X}}, [\beta(A^{\mathbb{X}}) \times \beta(A^{\mathbb{X}})], \nu = \mu \times \mu, F \times G)$ a dynamical system being their cartesian product. Then:

1. For any $n \in \mathbb{N} \setminus \{0\}$ $F^n : A^{\mathbb{X}} \to A^{\mathbb{X}}$, is a cellular automaton and $h(A^{\mathbb{X}}, F^n) = nh(A^{\mathbb{X}}, F)$, $h(F^n) = nh(F)$,

2. $[\beta(A^{\mathbb{X}}) \times \beta(A^{\mathbb{X}})] = \beta(A^{\mathbb{X}} \times A^{\mathbb{X}})$,

3. $F \times G : A^{\mathbb{X}} \times A^{\mathbb{X}} \to A^{\mathbb{X}} \times A^{\mathbb{X}}$ is surjective, continuous and preserves probabilistic measure ν,

4. $h(A^{\mathbb{X}}\times A^{\mathbb{X}},\, F\times G) = h(A^{\mathbb{X}},\, F)+h(A^{\mathbb{X}},\, G),\;\; h(\,F\times G) = h(F)+h(G)\,.$

Remark 1. *Non-empty and open set U, in $(A^{\mathbb{X}}\times A^{\mathbb{X}},\, \tau_{d'})$ is a sum of a countable family of cartesian products of balls from*

$$\alpha = \{K(x,\, r) : \, x\in A^{\mathbb{X}},\, r\in\mathbb{R},\, r>0\}\subset\tau_d$$

(α is countable base of the topology τ_d). Any cartesian product of two balls from α is an element of σ-algebra $\gamma = [\beta(A^{\mathbb{X}})\times\beta(A^{\mathbb{X}})]$. Just from the definition of a σ-algebra it follows that any sum of a countable family of sets from γ is in γ and so $U\in\gamma$. It implies that $\tau_{d'}\subset\gamma = [\beta(A^{\mathbb{X}})\times\beta(A^{\mathbb{X}})]$. From the facts $\gamma\subset\beta(A^{\mathbb{X}}\times A^{\mathbb{X}})$, $\tau_{d'}\subset\gamma$ and from the definition of the σ-algebra $\beta(A^{\mathbb{X}}\times A^{\mathbb{X}})$ it follows that $\gamma\supset\beta(A^{\mathbb{X}}\times A^{\mathbb{X}})$. Hence the equality $[\beta(A^{\mathbb{X}})\times\beta(A^{\mathbb{X}})] = \beta(A^{\mathbb{X}}\times A^{\mathbb{X}})$ is true. All the remaining statements of the above theorem are widely known [4, 7] from the theory of dynamical systems.

Let $\eta : \, A\to A$ be a permutation defined on A. A support of η is the set $supp(\eta) = \{a\in A : \, \eta(a)\neq a\}$. If $k\in\mathbb{N}\setminus\{0\}$ is the minimal number such that $\eta^k(a) = a$ for some fixed $a\in supp(\eta)$, and $\#supp(\eta) = k$, then the permutation η is said to be a cycle with length k. We finish this section by the following property of permutations.

Theorem 8. *If $\eta : \, A\to A$ is a permutation, not equal to identity $id_{|A}$, then there exists a finite set of cycles $\{\eta_i : \, A\to A\}_{i\in\{1,\,2,\,\ldots,\,m\}}$ such that:*
(a) $\eta = \eta_1\circ\eta_2\circ\ldots\circ\eta_m$,
(b) $supp(\eta_i)\cap supp(\eta_j) = \emptyset$ for any $i,\, j\in\{1,\, 2,\, \ldots,\, m\}$ and $i\neq j$.

4. Non positively expansive cellular automata for which $h(A^{\mathbb{X}},\, F) = h(F)\, > 0$

In this section we present some general constructions of cellular automata defined on metric spaces $(A^{\mathbb{N}},\, d)$ and $(A^{\mathbb{Z}},\, d)$ endowed with the uniform Bernoulli measure μ which are surjective, not positively expansive and μ is the measure of the maximal and positive entropy.

It is widely known that a cellular automaton given by a shift mapping $\sigma : \, A^{\mathbb{Z}}\to A^{\mathbb{Z}}$ is surjective, sensitive to initial conditions and not positively expansive. It is possible to define a local rule for this automaton in such a way that it is right-permutative. In the first example presented below we give a method of construction of surjective cellular automata defined on $(A^{\mathbb{Z}},\, d)$

which are right-permutative, sensitive to initial conditions and not positively expansive.

EXAMPLE 2. *Let* $\delta' : A \to A$ *be a permutation not equal to identity* $id_{|A}$. *Let* $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *be a cellular automaton given by the following local rule* $f' : A^3 \to A$, $f'(wa) = \delta'(a)$ *for any* $w \in A^2$, $a \in A$ *and* $f(x)(i) = f'(x_{[i-1,\, i+1]})$ *for any* $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$.
We claim that the defined cellular automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is surjective, sensitive to initial conditions, not positively expansive and such that* $h(A^{\mathbb{Z}}, f) = h(f) = \log \# A > 0$. *Additionally* $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is not conjugate to a cellular automaton given by a shift mapping* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$.

PROOF. Just from the theorem 8 for $\delta' : A \to A$ there exists a finite set of cycles $\{\delta_i : A \to A\}_{i \in \{1, 2, ..., m\}}$ such that $\delta' = \delta_1 \circ \delta_2 \circ ... \circ \delta_m$ and $supp(\delta_i) \cap supp(\delta_j) = \emptyset$ for any $i, j \in \{1, 2, ..., m\}$, $i \neq j$. For any cycle $\delta_i : A \to A$ it is possible to compute the minimal number $n_i \in \mathbb{N} \setminus \{0\}$ such that $\delta_i^{n_i}(a) = a$ for any fixed $a \in supp(\delta_i)$. Denote by N the minimal common multiple of the numbers $n_1, n_2, ..., n_m$. Hence N is the minimal number such that $\delta'^N = id_{|A}$.

Putting $\delta(x)(i) = \delta'(x(i))$ for any $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we expand the permutation $\delta' : A \to A$ to the mapping $\delta : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$. Note that $\delta'^N = id_{|A}$ implies $\delta^N = id_{|A^{\mathbb{Z}}}$. Hence $\delta : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton and according to the theorem 1 the following commutation is true $\delta\sigma = \sigma\delta$.

Just from the definition of the cellular automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ it follows that it is right-permutative and surjective (Theorem 4). Since $f = \delta \circ \sigma$ we have $f^N = (\delta \circ \sigma)^N = \delta^N \circ \sigma^N = id_{|A^{\mathbb{Z}}} \circ \sigma^N = \sigma^N$. Now Theorem 7 implies equalities: $h(A^{\mathbb{Z}}, f^N) = h(A^{\mathbb{Z}}, \sigma^N)$, $h(A^{\mathbb{Z}}, f^N) = Nh(A^{\mathbb{Z}}, f)$, $h(A^{\mathbb{Z}}, \sigma^N) = Nh(A^{\mathbb{Z}}, \sigma) = N \log \# A$.
Finally $Nh(A^{\mathbb{Z}}, f) = Nh(A^{\mathbb{Z}}, \sigma) = N \log \# A$ and $h(A^{\mathbb{Z}}, f) = \log \# A$. In a similar way we compute measure-theoretic entropy $h(f) = \log \# A$. Hence f is not a shift mapping on $A^{\mathbb{Z}}$ and $h(A^{\mathbb{Z}}, f) = h(f) = \log \# A > 0$. The fact that $f^N = \sigma^N$, and $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a mapping sensitive to initial conditions and not positively expansive implies the same properties for $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$. This finishes the proof. $\square$

It is a known fact (compare [2, 7]) that the topological entropy and measure-theoretic entropy of any equicontinuous surjective cellular automaton defined on the space $(A^{\mathbb{N}}, d)$ are equal 0. Hence the uniform Bernoulli measure is a measure of the maximal entropy of such a system. In the second example presented below we give a method of construction of one-sided surjective cellular automata which are not positively expansive and have positive and equal topological and measure-theoretic entropies. Thus it is a one more interesting case when the uniform Bernoulli measure is a measure of the maximal entropy.

EXAMPLE 3. *Let* $B = \{0, 1, 2, 3\}$. *Let us consider a cellular automaton* $F : B^{\mathbb{N}} \to B^{\mathbb{N}}$, *given by the following local rule* $F' : B^2 \to B$

$$F'(00) = F'(01) = F'(20) = F'(21) = 0,$$

$$F'(02) = F'(03) = F'(22) = F'(23) = 2,$$

$$F'(10) = F'(11) = F'(30) = F'(31) = 1,$$

$$F'(12) = F'(13) = F'(32) = F'(33) = 3,$$

$$F(z)(i) = F'(z_{[i,\, i+1]}) \textit{ for any } z \in B^{\mathbb{N}} \textit{ and } i \in \mathbb{N}.$$

We claim that the defined cellular automaton $F : B^{\mathbb{N}} \to B^{\mathbb{N}}$ *is surjective, not positively expansive and such that* $h(B^{\mathbb{N}}, F) = h(F) = \log \frac{\#B}{2} > 0$.

PROOF. For $A = \{0, 1\}$ consider cellular automata $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ and $I : A^{\mathbb{N}} \to A^{\mathbb{N}}$, given by the local rules

(a) $\sigma' : A^2 \to A$, where $\sigma'(00) = \sigma'(10) = 0$, $\sigma'(01) = \sigma'(11) = 1$, and $\sigma(x)(i) = \sigma'(x_{[i,\, i+1]})$ for any $x \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$,

(b) $I' : A^2 \to A$, where $I'(00) = I'(01) = 0$, $I'(10) = I'(11) = 1$, and $I(x)(i) = I'(x_{[i,\, i+1]})$ for any $x \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$.

Note that for any $x \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$, $\sigma(x)(i) = x(i+1)$, what means that the cellular automaton $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is surjective, defines a shift mapping, is positively expansive, $h(A^{\mathbb{N}}, \sigma) = \log \#A > 0$, and by Theorem 3.8 $h(A^{\mathbb{N}}, \sigma) = h(\sigma)$. Observe also that the automaton $I : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is identity, surjective, equicontinuous and $h(A^{\mathbb{N}}, I) = h(I) = 0$.

The above automata are associated with the following symbolic dynamical systems:

(a) $(A^{\mathbb{N}}, \beta(A^{\mathbb{N}}), \mu, \sigma)$,

(b) $(A^{\mathbb{N}}, \beta(A^{\mathbb{N}}), \mu, I)$.

Let us introduce a metric space $(A^{\mathbb{N}} \times A^{\mathbb{N}}, d')$, where $d'(z, z') = max\{d(x, x'),\ d(y, y')\}$ for any $z = (x, y)$, $z' = (x', y') \in A^{\mathbb{N}} \times A^{\mathbb{N}}$ and d is the original metric of $A^{\mathbb{N}}$. Now consider a dynamical system $(A^{\mathbb{N}} \times A^{\mathbb{N}}, [\beta(A^{\mathbb{N}}) \times \beta(A^{\mathbb{N}})], \nu = \mu \times \mu, f = \sigma \times I)$, given by the cartesian product of the defined symbolic dynamical systems (a), (b).

In view of the Theorem 7 $f = \sigma \times I : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}} \times A^{\mathbb{N}}$ is surjective and $h(A^{\mathbb{N}} \times A^{\mathbb{N}}, f = \sigma \times I) = h(f = \sigma \times I) = \log \#A > 0$. For $\overline{z} = (0^\omega, 0^k 10^\omega)$, $z = (0^\omega, 0^\omega)$, it holds

$$d'(f^n(\overline{z}), f^n(z)) = max\{d(\sigma^n(0^\omega), \sigma^n(0^\omega)), d(I^n(0^k10^\omega), I^n(0^\omega))\} = d(I^n(0^k10^\omega), I^n(0^\omega)) = d(0^k10^\omega, 0^\omega).$$

Of course the automaton $I : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is equicontinuous and $\forall \varepsilon > 0$, $\exists k \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $d'(f^n(\overline{z}), f^n(z)) = d(0^k10^\omega, 0^\omega) < \varepsilon$. Hence $f : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}} \times A^{\mathbb{N}}$ is not positively expansive.

Let $B = A^2 = \{00, 01, 10, 11\}$. The metric space $(B^{\mathbb{N}}, d_B)$ is introduced in exactly the same way as $(A^{\mathbb{N}}, d)$. Having $(B^{\mathbb{N}}, d_B)$ it is possible to consider the symbolic dynamical system $(B^{\mathbb{N}}, F)$ associated with cellular automaton $F : B^{\mathbb{N}} \to B^{\mathbb{N}}$, given by the following local rule $F' : B^2 \to B$:

(1) $F'(abcd) := \sigma'(ac)I'(bd)$ for any $ab,\, cd \in B$, $a, b, c, d \in A$,

(2) $F(z)(i) = F'(z_{[i,\, i+1]})$ for any $z \in B^{\mathbb{N}}$ and $i \in \mathbb{N}$.

We will show that dynamical systems $(A^{\mathbb{N}} \times A^{\mathbb{N}}, f = \sigma \times I)$, $(B^{\mathbb{N}}, F)$ are conjugate and that the conjugate mapping $s : A^{\mathbb{N}} \times A^{\mathbb{N}} \to B^{\mathbb{N}}$ is of the form $s(x, y)(i) = x(i)y(i) \in B$ for any $x, y \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$. We also prove that $(A^{\mathbb{N}} \times A^{\mathbb{N}}, [\beta(A^{\mathbb{N}}) \times \beta(A^{\mathbb{N}})], \nu, f)$, $(B^{\mathbb{N}}, \beta(B^{\mathbb{N}}), \mu_B, F)$ are measurably isomorphic.

From the theory of dynamical systems [4, 7] it is known that the following conditions are sufficient for the above two systems to be conjugate (topologically isomorphic) and measurably isomorphic:

(1) $s : A^{\mathbb{N}} \times A^{\mathbb{N}} \to B^{\mathbb{N}}$ is bijection, s is continuous (that is $s^{-1}(Q) \in \tau_{d'}$ for any $Q \in \tau_{d_B}$), and $s \circ f = F \circ s$,

(2) s^{-1}, s are measurable (that is $s(P) \in \beta(B^{\mathbb{N}})$, $s^{-1}(Q) \in [\beta(A^{\mathbb{N}}) \times \beta(A^{\mathbb{N}})]$) and measure-preserving (that is $\nu(s^{-1}(Q)) = \mu_B(Q)$, $\mu_B(s(P)) = \nu(P)$) for any $P \in [\beta(A^{\mathbb{N}}) \times \beta(A^{\mathbb{N}})]$, $Q \in \beta(B^{\mathbb{N}})$.

Notice that the first condition is sufficient for the above two systems to be conjugate [7].

It is clear that the mapping $s : A^{\mathbb{N}} \times A^{\mathbb{N}} \to B^{\mathbb{N}}$ is bijective and continuous. This implies that its inverse $s^{-1} : B^{\mathbb{N}} \to A^{\mathbb{N}} \times A^{\mathbb{N}}$ is also continuous [7], so $s : A^{\mathbb{N}} \times A^{\mathbb{N}} \to B^{\mathbb{N}}$ is a homeomorphism. Moreover, for any $x, y \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$ we have

$$\begin{aligned}(s \circ f)(x, y)(i) = {} & s(\sigma(x), I(y))(i) = \\ & \sigma(x)(i)I(y)(i) = \sigma'(x(i)(x+1))I'((y(i)y(i+1)) = \\ & F'(x(i)y(i)x(i+1)y(i+1)) = F'(s(x, y)(i)s(x, y)(i+1)) = \\ & (F \circ s)(x, y)(i).\end{aligned}$$

This proves the equality $s \circ f(x,y) = F \circ s(x, y)$ for any $x, y \in A^{\mathbb{N}}$. Hence dynamical systems $(A^{\mathbb{N}} \times A^{\mathbb{N}}, f = \sigma \times I)$, $(B^{\mathbb{N}}, F)$ are conjugate.

It follows from the Theorem 7 that $[\beta(A^{\mathbb{N}}) \times \beta(A^{\mathbb{N}})] = \beta(A^{\mathbb{N}} \times A^{\mathbb{N}})$. The mapping s is a homeomorphism and so s, s^{-1} are measurable [8]. Hence to prove that $(A^{\mathbb{N}} \times A^{\mathbb{N}}, \beta(A^{\mathbb{N}} \times A^{\mathbb{N}}), \nu = \mu \times \mu, f)$, $(B^{\mathbb{N}}, \beta(B^{\mathbb{N}}), \mu_B, F)$ are measurably isomorphic we show firstly that

$\mu_B(s(K((x, y), 2^{-n})) = \nu(K((x, y), 2^{-n}))$ for any ball $K((x, y), 2^{-n}) \in \beta(A^{\mathbb{N}} \times A^{\mathbb{N}})$,

where $(x, y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$, $n \in \mathbb{N}$.

Notice that

$$\nu(K((x, y), 2^{-n})) = \nu(K(x, 2^{-n}) \times K(y, 2^{-n})) = \mu(K(x, 2^{-n})) \cdot \mu(K(x, 2^{-n})) = \#A^{-(n+1)} \cdot \#A^{-(n+1)} = \#A^{-2(n+1)}, \ \mu_B(s(K((x, y), 2^{-n}))) = \mu_B(K(z, 2^{-n})) = \#B^{-(n+1)} = \#A^{-2(n+1)}, \text{ where } z = s(x, y) \in B^{\mathbb{N}}, n \in \mathbb{N}.$$

The family $\beta = \{K((x, y), r)\} : (x, y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}, \ r \in \mathbb{R}, \ r > 0\}$ is the base of $\tau_{d'}$. Any two balls in $(A^{\mathbb{N}} \times A^{\mathbb{N}}, d')$ are disjoint or one is contained in the other. Since s is a homeomorphism then any non-empty and open set W, in $(A^{\mathbb{N}} \times A^{\mathbb{N}}, \tau_{d'})$ (and also open set $s(W)$, in $(B^{\mathbb{N}}, \tau_{d_B})$) is a sum of countable family of pairwise disjoint balls. According to the additivity of probabilistic measures ν, μ_B, if $W \in \tau_{d'}$, then $\mu_B(s(W)) = \nu(W)$. It is true that if $P \in \beta(A^{\mathbb{N}} \times A^{\mathbb{N}})$, then $s(P) \in \beta(B^{\mathbb{N}})$. We also know that ν is in $(A^{\mathbb{N}} \times A^{\mathbb{N}}, \beta(A^{\mathbb{N}} \times A^{\mathbb{N}}))$ a probabilistic and regular Borel measure [8]. It means that for $P \in \beta(A^{\mathbb{N}} \times A^{\mathbb{N}})$ and any $\varepsilon > 0$ there exist sets $U, V \in \tau_{d'}$ such that $K = (A^{\mathbb{N}} \times A^{\mathbb{N}}) \setminus V$, $K \subset P \subset U$ and $\nu(U) \setminus \nu(K) = \nu(U \setminus K) < \varepsilon$. We have

$$\mu_B(s(K)) = \mu_B(s((A^{\mathbb{N}} \times A^{\mathbb{N}}) \setminus V)) = \mu_B(s(A^{\mathbb{N}} \times A^{\mathbb{N}})) \setminus \mu_B(s(V)) = \nu(A^{\mathbb{N}} \times A^{\mathbb{N}}) \setminus \nu(V) = \nu((A^{\mathbb{N}} \times A^{\mathbb{N}}) \setminus V) = \nu K.$$

The fact $K \subset P \subset U$, implies that

$$s(K) \subset s(P) \subset s(U), \ \nu(K) = \mu_B(s(K)) \leq \mu_B(s(P)) \leq \mu_B(s(U)) = \nu(U).$$

Thus $\mu_B(s(P)) = \nu(P)$ for any $P \in \beta(A^{\mathbb{N}} \times A^{\mathbb{N}})$.

We conclude that the cellular automaton $F = s \circ f \circ s^{-1} : B^{\mathbb{N}} \to B^{\mathbb{N}}$ is surjective, not positively expansive and such that

$$h(B^{\mathbb{N}}, F) = h(A^{\mathbb{N}} \times A^{\mathbb{N}}, f = \sigma \times I),$$

$$h(F) = h(f = \sigma \times I) \text{ and } h(B^{\mathbb{N}}, F) = h(F) = \log \tfrac{\#B}{2} = \log \#A > 0.$$

In a similar way we prove that $\nu(s^{-1}(Q)) = \mu_B(Q)$ for any $Q \in \beta(B^{\mathbb{N}})$.

Hence the cellular automaton $F = s \circ f \circ s^{-1} : B^{\mathbb{N}} \to B^{\mathbb{N}}$ is surjective, not positively expansive and such that $h(B^{\mathbb{N}}, F) = h(A^{\mathbb{N}} \times A^{\mathbb{N}}, f = \sigma \times I)$, $h(F) = h(f = \sigma \times I)$ and $h(B^{\mathbb{N}}, F) = h(F) = \log \frac{\#B}{2} = \log \#A > 0$.

Encode A^2 as follows $00 =: 0, 01 =: 1, 10 =: 2, 11 =: 3$ to obtain the new alphabet $B = \{0, 1, 2, 3\}$. Taking into account the new alphabet $B = \{0, 1, 2, 3\}$ and the definition of F' we see that the mapping F' is exactly the same as defined at the beginning of the example. Hence the obtained conclusions can be applied to the cellular automaton $F : B^{\mathbb{N}} \to B^{\mathbb{N}}$ defined at the beginning of the example. This finishes the proof. $\square$

5. References

[1] Amoroso S., Patt Y.N.; *Decision Procedures for Surjectivity and Injectivity of Parallel Maps for Tessellation Structures*, Journal of Computer and System Sciences, 6, 1972, pp. 448–464.

[2] Blanchard F., Kurka P., Maass A.; *Topological and Measure-Theoretic Properties of One-Dimensional Cellular Automata*, Physica D, 103, 1997, pp. 86–89.

[3] Blanchard F., Maass A.; *Dynamical Properties of Expansive Cellular Automata*, Israel Journal of Mathematics, 99, 1997, pp. 149–174.

[4] Brin M., Stuck G.; *Introduction to Dynamical Systems*, Cambridge University Press, 2002.

[5] Goodwyn L.W.; *Topological Entropy Bounds Measure-Theoretic Entropy*, Proc. Amer. Math. Soc., 23, 1969, pp. 679–688.

[6] Kurka P.; *Languages, Equicontinuity and Attractors in Cellular Automata*, Ergodic Theory Dynamical Systems, 17, 1997, pp. 417–433.

[7] Kurka P.; *Topological and Symbolic Dynamics*, SMF, Cours Specialises, 2003.

[8] Rudin W.; *Real and Complex Analysis*, McGraw-Hill, 1966.

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